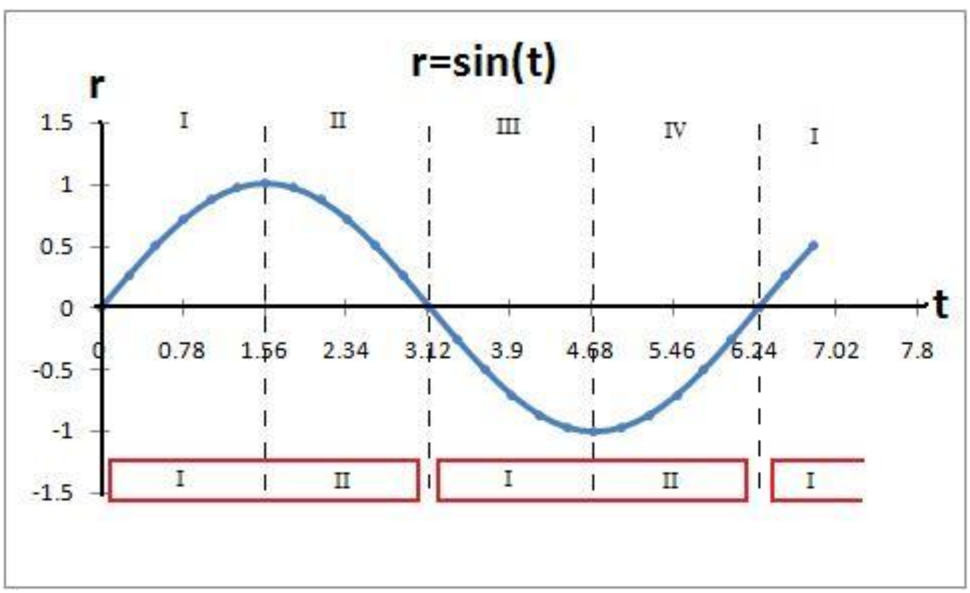
I think I have finally found an answer to a problem that has been bothering me for years. As a teacher, I try to have my students memorize as few facts as possible for the simple reason that they may forget them (especially when under stress or time pressure, such as during an AP test). If a fact is not absolutely fundamental, I greatly prefer that they be able to derive it quickly and correctly rather than memorize it. However, when working with polar functions $r = f(\theta)$, this philosophical decision has led to some difficulties. In particular, my students have oftentimes been unsure of what domain to choose so that the figure has enough time to "complete itself" – for example, a function such as $r = \cos(\theta)$ completes itself by the time $\theta = \pi$, whereas $r = \sin(\frac{1}{2}\theta)$ doesn't complete itself until $\theta = 4\pi$. I know there is a (rather complicated) set of function-specific rules that they could memorize, but I find the utility of these rules to be limited to this topic. I therefore do not find memorization of these rules to be a worthwhile endeavor. We have instead relied upon trial and error in the past to determine an appropriate domain.

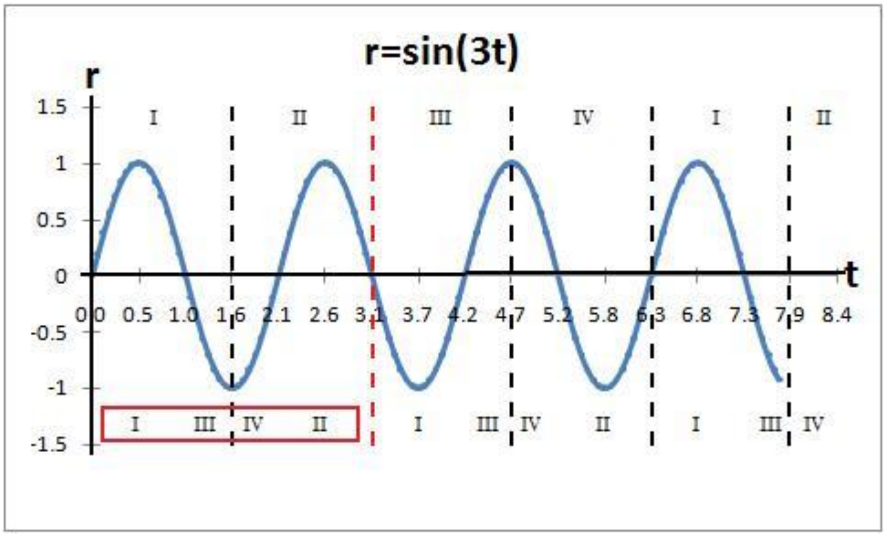
Recently, however, I have stumbled upon a rather simple method to determine an appropriate domain when graphing polar functions, something which has eluded me for quite some time. This method is based on finding the "real" quadrant that one will be using at any given time. For example, if $\theta = \frac{2\pi}{3}$ and $r > 0$, the point will lie in quadrant II, and thus the "real" quadrant will be II. If, however, $\theta = \frac{2\pi}{3}$ and $r < 0$, the point will actually lie in quadrant IV, not quadrant II as might be suggested by the value of θ. Thus the "real" quadrant is IV.

Consider the analysis of $r = \sin(\theta)$ below. Along the top of the graph are the quadrants, and along the bottom of the graph are the "real" quadrants.



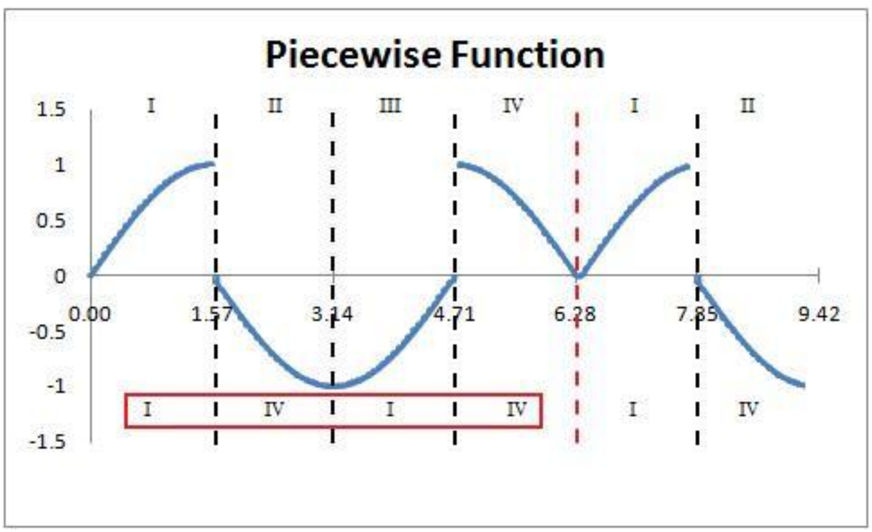
Note that after $\theta = \pi$, the "real" quadrants begin repeating themselves. When θ is in quadrant III, the "real" quadrant is I. By the symmetry of the sine curve, we can see that these points will map on to the points we have already graphed when θ was in quadrant I since we are now only considering the magnitude of r. Similarly, when θ is in quadrant IV, the "real" quadrant is II. These points will map on to the points we already graphed when we were in quadrant II. Thus, we will only need to consider a domain of $\theta \in [0, \pi]$ when we graph this function on the Cartesian Plane.

Here is a more complex illustration of this method:



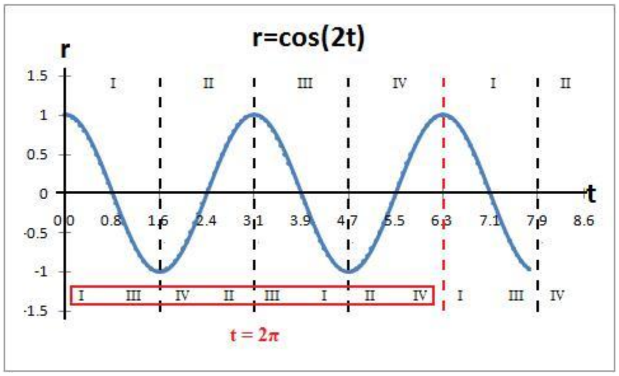
The cycle of "real" quadrants repeats itself after $\theta = \pi$. Since the magnitude of r increases from 0 to 1 for θ in both $[0, \frac{\pi}{6}]$ and $[\pi, \frac{7\pi}{6}]$, these intervals must be considered identical with respect to the Cartesian graph. Thus, we select a domain of $\theta \in [0, \pi]$ when graphing on the Cartesian Plane.

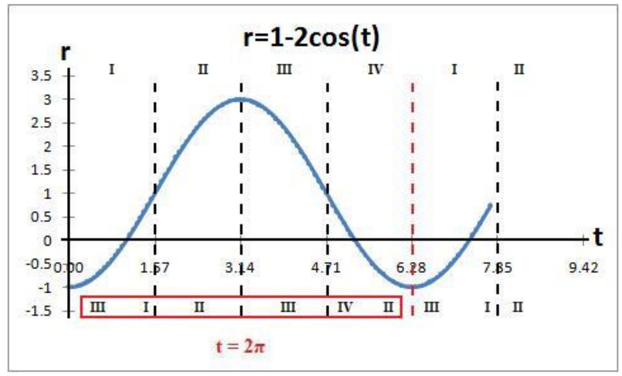
Finally, note that the repetition of the cycle of "real" quadrants is a necessary but not sufficient condition. Consider the following piecewise defined function (suggested by Alan Lipp) which has a period of 2π:

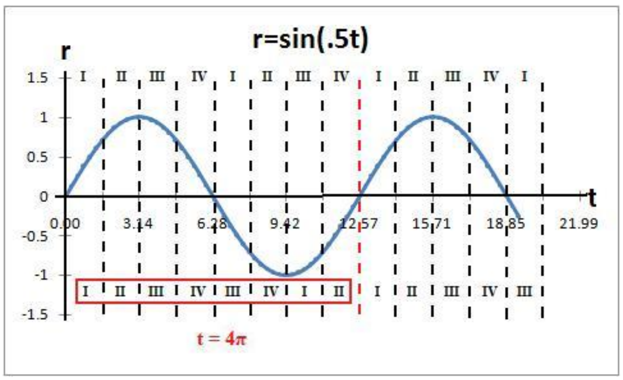


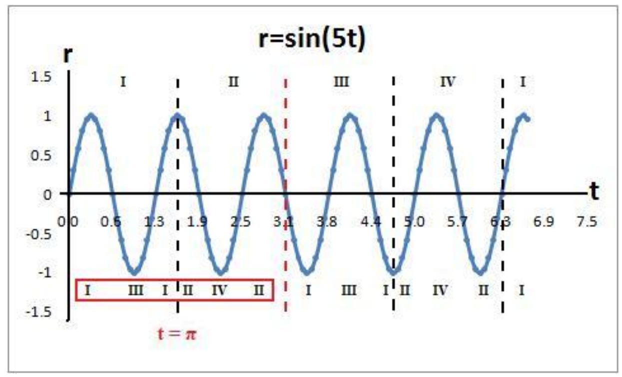
It is true that the cycle of "real" quadrants repeats itself after $\theta = \pi$. However, it must also be noticed that the behavior for $\theta \in [0, \pi]$ is not the same as that for $\theta \in [\pi, \frac{3\pi}{2}]$. In the first interval, the magnitude of r increases from 0 to 1, while in the second the magnitude of r decreases from 1 to 0, thus the points on the graph from these intervals cannot possibly map on to each other. For this reason, it is not correct to assume that the minimal interval is $[0, \pi]$. That said, note the interval $[2\pi, \frac{5\pi}{2}]$. Here we see the "real" quadrant is also I. Additionally, note the magnitude of r increases from 0 to 1 in this interval, just as it does for $[0, \frac{\pi}{2}]$. Thus the first repetition of the cycle must be rejected due to the lack of the requisite symmetry, and the domain $\theta \in [0, 2\pi]$ must be chosen instead.

Here are a few more examples:









I think this may prove to be a highly efficient way for my students to eliminate some of the guesswork from their graphing without requiring any tedious memorization, especially since we are already graphing r versus θ to generate our Cartesian graph. This method has the considerable added benefit of focusing their minds on the proper quadrants when $r < 0$. Rather than having to perform the mental gymnastics required for working with negative values of r directly, they can go straight to the quadrant they will actually be graphing in and use the magnitude of r instead. When we turn our attention to the Calculus of polar functions (in particular, areas of regions enclosed by polar functions), we will be able to find appropriate limits of integration readily, and in some instances we will not even need to consider the Cartesian Plane.

If anyone spots a flaw in this method, please do not hesitate to contact me. I am aware that it is limited by the need for symmetry for its conclusions to be valid, but beyond that I believe it to generalize widely. If you are aware of a pathological case that causes failure or if you have any other feedback, I would love to hear about it (but please realize that for the purposes of my students, we will be working with relatively tame functions such as those in the examples above). I can be contacted at steven dot p dot hiller at gmail dot com; be sure to include the word “polar” in the subject line and forgive my use of spambot defenses in the email address above!