فصل پنجم

ضد مشتق

9-2-5 تمرین صفحه 350

(1) هر یک از انتگرال های زیر را حل کنید:

الف)
$$\int (3x-2)^4 dx = \frac{1}{3}\int (3x-2)^4 3dx = \frac{1}{3}\cdot\frac{(3x-2)^5}{5}+c$$

ب)
$$\int \frac{x+1}{2\sqrt{x+1}}dx = \frac{1}{2}\int \sqrt{x+1}dx = \frac{1}{2}\times\frac{2}{3}(x+1)\sqrt{x+1}+c$$

ج)
$$\int x^2\sqrt{1+x}dx$$

قرار دهید $u^2 = 1+x$ پس $2udu = dx$

$$\int (u^2-1)^2 2u^2 du = 2\int (u^6-2u^4+u^2)du$$
$$=2\left(\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}\right)+c$$
$$=2\left(\frac{(x+1)^{\frac{7}{2}}}{7}-\frac{2(x+1)^{\frac{5}{2}}}{5}+\frac{(x+1)^{\frac{3}{2}}}{3}\right)+c$$

د)
$$\int x^3\sqrt{x^2-1}dx$$
$$u^2 = x^2-1 \Rightarrow 2udu = 2xdx \Rightarrow udu = xdx$$
$$\int (u^2+1)udu = \frac{u^4}{4}+\frac{u^2}{2}+c = \frac{(x^2-1)^2}{4}+\frac{x^2-1}{2}+c$$

ه)
$$\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx$$

$$u=\sqrt{x}\Rightarrow 2du=\frac{dx}{\sqrt{x}}\Rightarrow 2\int(u-1)^2du=\frac{2}{3}(u-1)^3+c$$
$$=\frac{2}{3}(\sqrt{x}-1)^3+c$$

$$\int\frac{x^5dx}{\sqrt{1-x^6}}=-\frac{1}{6}\int\frac{-6x^5}{\sqrt{1-x^6}}dx=-\frac{1}{6}\times 2\sqrt{1-x^6}+c$$
$$=-\frac{1}{3}\sqrt{1-x^6}+c$$

و)

$$\int\frac{xdx}{\sqrt{1+x^2\sqrt{(1+x^2)^3}}}=\int\frac{xdx}{\sqrt{1+x^2(1+x^2)\sqrt{1+x^2}}}$$
$$u^2=1+x^2\Rightarrow udu=xdx$$
$$\int\frac{udu}{\sqrt{1+(u^2-1)u^3}}=\int\frac{udu}{\sqrt{1-u^3+u^5}}$$

ز)

$$\int\frac{(x+1)dx}{(x^2+2x+2)^3}=\int\frac{(x+1)dx}{((x+1)^2+1)^3}$$
$$u=(x+1)^2+1\Rightarrow\frac{du}{u^3}=\frac{1}{2}\times-\frac{1}{2}\times\frac{1}{u^2}+c=-\frac{1}{4((x+1)^2+1)^2}+c$$

ح)

$$\int\frac{\sqrt{4-x^2}}{x^4}dx;u=2\sin x\Rightarrow du=2\cos xdx$$
$$\frac{1}{4}\int\frac{\cos^2 x}{\sin^4 x}dx=\frac{1}{4}\int\cot^2 x\cdot\csc^2 xdx=-\frac{1}{12}\cot^3 x+c$$

ط)

ی)

$$\int \frac{x^2+1}{\sqrt[3]{x^3+3x+1}}dx$$

$$u = x^3+3x+1 \Rightarrow \frac{du}{3} = (x^2+1)dx$$

$$\frac{1}{3}\int \frac{du}{\sqrt[3]{u}} = \frac{1}{3}\times\frac{3}{2}u^{\frac{2}{3}} + c = \frac{1}{2}(x^3+3x+1)^{\frac{2}{3}} + c$$

ک)

$$\int x^2\sqrt[3]{1-x}dx$$

$$u^3 = 1-x \Rightarrow -3u^2du = dx$$

$$-3\int (1-u^3)^2u^3du = -3\int (u^3-2u^6+u^9)du$$

$$= -3(\frac{u^4}{4} - \frac{2u^7}{7} + \frac{u^{10}}{10}) + c$$

$$= -3(\frac{(1-x)^{\frac{4}{3}}}{4} - \frac{2(1-x)^{\frac{7}{3}}}{7} + \frac{(1-x)^{\frac{10}{3}}}{10}) + c$$

ل)

$$\int \frac{x^2}{\sqrt[3]{x^3+1}}dx = \frac{1}{3}\int \frac{3x^2}{\sqrt[3]{x^3+1}}dx = \frac{1}{3}\times\frac{3}{2}(x^3+1)^{\frac{2}{3}} + c$$

م)

$$\int x\sqrt[5]{5-x^2}dx = -\frac{1}{2}\int \sqrt[5]{5-x^2}(-2x)dx$$

$$= -\frac{1}{2}\times\frac{5}{6}(5-x^2)^{\frac{6}{5}} + c$$

(2) فرض کنید $f(x) = |x|$ و تابع F به صورت زیر تعریف شده باشد:

$$F(x) = \begin{cases} -\frac{1}{2}x^2, & x < 0 \\ \frac{1}{2}x^2, & x \geq 0 \end{cases}$$

نشان دهید F یک ضدّ مشتق f روی $(-\infty,+\infty)$ است.

حل)

$$F'(x)=\begin{cases}-x & , \quad x<0\\ x & , \quad x\geq 0\end{cases}=f(x)$$

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انتگرال $\int\frac{dx}{\sin^2 x\cos^4 x}$ را حل کنید.

حل)

$$\int\frac{dx}{\sin^2 x\cos^4 x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cdot\cos^4 x}dx=\int\frac{dx}{\cos^4 x}+\int\frac{dx}{(\sin x\cos x)^2}$$

$$=\int\sec^4 xdx+4\int\frac{dx}{(\sin 2x)^2}=\int\sec^2 x(1+\tan^2 x)dx+4\int\csc^2 2x$$

$$=\tan x+\frac{\tan^3 x}{3}-2\cot 2x+c$$

20-3-5 تمرین صفحه 357 .

(1) هر یک از انتگرال های زیر را حساب کنید.

1)

$$\int\tan^4 xdx=\int\tan^2 x(\sec^2 x-1)dx$$

$$=\int\tan^2 x\sec^2 xdx-\int(\tan^2 x+1)dx+\int dx$$

$$=\frac{\tan^3 x}{3}-\tan x+x+c$$

2)

$$\int\tan^6 xdx=\int\tan^2 x\cdot\tan^4 xdx=\int(\sec^2 x-1)\tan^4 xdx$$

$$=\int\tan^4 x\cdot\sec^2 xdx-\int\tan^4 xdx$$

$$=\frac{\tan^5 x}{5}-\frac{\tan^3 x}{3}+\tan x-x+c$$

$$\int \frac{\cos^3 x}{\sin^9 x} dx$$
$$= \int \frac{\cos x(1-\sin^2 x)}{\sin^9 x} dx = \int \frac{\cos x}{\sin^9 x} dx - \int \frac{\cos x}{\sin^7 x} dx$$
$$= -\frac{1}{8\sin^8 x} + \frac{1}{6\sin^6 x} + c$$

(3

$$\int \sec^4 x \cdot \cot^6 x dx = \int \frac{\sec^4 x}{\tan^6 x} dx = \int \frac{\sec^2 x(1+\tan^2 x)}{\tan^6 x} dx$$
$$= \int \frac{\sec^2 x}{\tan^6 x} dx + \int \frac{\sec^2 x}{\tan^4 x} dx = -\frac{1}{5\tan^5 x} - \frac{1}{3\tan^3 x} + c$$

(4

5) $$\int \frac{\sin^3 x}{\cos^9 x} = \int \frac{\sin x(1-\cos^2 x)}{\cos^9 x}dx = \int \frac{\sin x}{\cos^9 x}dx - \int \frac{\sin x}{\cos^9 x}dx$$
$$= \frac{1}{8\cos^8 x} - \frac{1}{6\cos^6 x} + c$$

6) $$\left|\csc^4 x.\cot^7 x dx - \right|\csc^2 x(\cot^2 x - 1)\cot^7$$
$$= \int \cot^9 x.cse^2 xdx - \int \cot^7 x.cse^2 xd\ x$$
$$= -\frac{\cot^{10} x}{10} + \frac{\cot^8 x}{8} + c$$

7) $$\int \cos x \sec^2(\sin x)dx = \int \frac{\sin x \cot x}{\cos^2 x}dx$$
$$= \frac{1}{2}\int \frac{2\sin x\cos x}{\cos^2 x}dx = -\frac{1}{2\cos x} + c$$

8) $$\int \sin^3 2x.\cos^5 2x\,dx = \int \sin 2x(1-\cos^2 2x)\cos^5 2x\,dx$$
$$= \int \cos^5 2x.\sin 2x\,dx - \int \cos^7 2x.\sin 2x\,dx$$
$$= -\frac{1}{12}\cos^6 2x + \frac{1}{16}\cos^8 2x + c$$

9) $$\int \frac{\tan x(1+\tan x)^{10}}{\cos^2 x}dx = |u(1+u)^{10}du$$
$$\left(u = \tan x \Rightarrow du = \frac{dx}{\cos^2 x}\right)$$
$$= \int\left((u+1)^{11} - (u+1)^{10}\right)du = \frac{(\tan x + 1)^{12}}{12} - \frac{(\tan x + 1)^{11}}{11}$$

10) $$\int \frac{\sin x\,dx}{\cos^2 x + 2\cos + 1} = \int \frac{\sin x\,dx}{(\cos x + 1)^2}$$
$$(u = \cos x + 1 \quad \Rightarrow \quad -du = \sin x\,dx)$$
$$= -\int \frac{du}{u^2} = \frac{1}{\cos x + 1} + c$$

(2) فرض کنید $In = |\tan^n x\,dx$. یک فرمول بازگشتی برای محاسبه In بیابید, I_4 و I_6 را محاسبه کنید.

حل)

$$In = \int \tan^{n-2} x \,.\, \tan^2 x\,dx = \int \tan^n x(\sec^2 x - 1)dx = \tan x$$
$$= \int \tan^{n-1} x \,.\, \tan^2 x\,dx - \int \tan^{n-2} x(\sec^{n-2} x - 1)dx$$
$$= \frac{\tan^{n-1}}{n-1} - I_{n-2}6 + c$$
$$\Rightarrow \qquad In + I_{n-2} = \frac{\tan^{n-1}}{n-1} + c$$
$$I_2 = \int \tan^3 x\,dx = \int (\sec^2 x - 1)dx = \tan x - x$$
$$I_4 = \frac{\tan^3 x}{3} - \tan x + x + c$$
$$I_6 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$$

(3) معادله دسته منحنی هایی رابیابید که ضریب زاویهخطوط مماس در هر نقطه (x, y)

از آن برابر $-\frac{\bar{x}}{y}$ باشد.

حل)

$$\frac{dy}{dx}=-\frac{x}{y} \Rightarrow ydy=-xdx$$

$$\Rightarrow \int ydy=-\int xdx+c$$

$$\frac{y^2}{2}=-\frac{x^2}{2}=c \Rightarrow x^2+y^2=2c$$

معادله دسته مخفی‌ها، دوایر بر مرکز مبدأ مختصات است.

(4) معادله مخفی را بیابید که از نقطه $(2\ ,\ 9)$ گذشته و معادله ضریب زاویه مماس برخی $3x^2$ باشد.

$$y=3x^2 \Rightarrow y=\int 3x^2dx+c$$

$$y=x^3+c$$

$$9=8+c \Rightarrow c=1$$

$$y=x^3+1$$

حل)

(5) معادله $y'-2x=0$ را حل کنید.

$$y'=2x \Rightarrow y=\int 2x+c$$

$$\Rightarrow =x^2+c$$

حل)

(6) مشتق تابعی برابر $\sqrt{x+3}$ است. هرگاه مقدار به ازای $x=1$ برابر 1 باشد, تابع را بیابید.

$$y'=\sqrt{x+3} \quad \Rightarrow \quad y=\int \sqrt{x+3}+c$$

$$\Rightarrow \quad y=\frac{2}{3}(x+3)\sqrt{x+3} \; + \; c$$

$$(1 \; , \; 1) \quad \Rightarrow \quad 1=\frac{2}{3}\times 4\times 2\times c \quad \Rightarrow \quad c=-\frac{13}{3}$$

$$y=\frac{2}{3}(x+3)\sqrt{x+3} \quad -\frac{13}{3}$$

حل)

(7) هر یک از انترالهای زیر را حل کنید.

1) $$\int \frac{\sin x\, dx}{(1+\cos x)^2}=-\frac{1}{1+\cos x}+c$$

$$\int \cos^6 x\, dx=\frac{1}{8}\int (1+\cos 2x)^3\, dx$$

2) $$=\frac{1}{8}\int \left(1+3\cos 2x+3\cos^2 2x+\cos^3 2x\right) dx$$

$$=\frac{1}{8}\left(\int dx+3\int \cos 2x\, dx+\frac{3}{2}\int (1+\cos 4x)dx+\int \cos 2x\left(1-\sin^2 2x\right) dx\right)$$

$$=\frac{1}{8}\left(x+\frac{3}{2}\sin 2x+\frac{3}{2}x+\frac{3}{8}\sin 4x+\frac{1}{2}\sin 2x-\frac{\sin^3 2x}{6}\right)+c$$

3) $$\int \sin^5 x\cos^2 x\, dx=\int \sin x\left(1-\cos^2 x\right)^2 \cos^2 x dx$$

$$=\int \sin x \, . \, \cos^2 x\; dx-2\int \sin x \; . \; \cos 4x\; dx+\int \sin x \; . \; \cos^6 x\; dx$$

$$=-\frac{\cos^3 x}{3}+\frac{2}{5}\cos^5 x-\frac{\cos^7 x}{7}+c$$

4) $$\int \frac{\cos^3 3x}{\sqrt[3]{\sin 3x}}\,dx = \int \cos 3x\left(1-\sin^2 3x\right)\sin^{-\frac{1}{3}} 3x\,dx$$

$$= \int \sin^{-\frac{1}{3}} 3x \,.\, \cos 3x\,dx - \int \sin^{\frac{5}{3}} x \,.\, \cos 3x\,dx$$

$$= \frac{1}{2}\sin^{\frac{2}{3}} 3x + \frac{1}{8}\sin^{\frac{8}{3}} 3x + c$$

5) $$\int \sin 3y \cos 3y \; dy = \frac{1}{6}\sin^2 3y + c$$

6) $$\int \cos t \,.\, \cos 3t \; dt = \frac{1}{2}\int\left(\cos 2t + \cos t\right)dt$$

$$= \frac{1}{4}\sin 2t + \frac{1}{2}\sin t + c$$

7) $$\int \sin x \,.\, \sin 3x \,.\, \sin 5x = \int \frac{1}{2}\left(\cos 2x - \cos x\right)\sin 5x\,dx$$

$$= \frac{1}{2}\int \sin 5x \,.\, \cos 2x\,dx - \frac{1}{2}\int \sin 5x \,.\, \cos x\,dx$$

$$= \frac{1}{2}\int\left(\sin\frac{7}{2}x + \sin\frac{3}{2}x\right)dx - \frac{1}{2}\int\left(\sin 3x + \sin 2x\right)dx$$

$$= \frac{-1}{7}\cos\frac{7}{2}x - \frac{1}{3}\cos\frac{3}{2}x + \frac{1}{6}\cos 3x + \frac{1}{4}\cos 2x + c$$

8) $$\int \sin^4 x \cos^4 x\, dx = \frac{1}{16}\int (2\sin x \cos x)^4 dx$$

$$= \frac{1}{16}\int \sin^4 2x\, dx = \frac{1}{32}\int (1-\cos 4x)^2$$

$$= \frac{1}{32}\int \left(1 - 2\cos 4x + \cos^2 4x\right) dx$$

$$= \frac{1}{32}\left(x - \frac{1}{2}\sin 4x + \frac{1}{2}x + \frac{1}{16}\sin 8x \right) + c$$

9) $$\int \sqrt{1+\sin^2(x-1)}\ .\ \sin(x-1)\cos(x-1)dx$$

$$\left(u = 1+\sin^2(x-1) \qquad \Rightarrow \qquad \frac{du}{2} = \sin(x-1)\cos(x-1)dx \right)$$

$$I = \frac{1}{2}\int \sqrt{u}\ \ du = \frac{1}{3}u\sqrt{u} + c = \frac{1}{3}\left(1+\sin^2(x-1)\right)\sqrt{1+\sin^{2(x-1)}} + c$$

10) $$\int \frac{dx}{\cos^2 x\ .\ \sin^2 x} = 4\int \frac{dx}{(2\sin x \cos x)^2} = 4\int \frac{dx}{\sin^2 2x} = -2\cot 2x + c$$

11) $$\int \sqrt{-+\sin^2(x-1)} \qquad .\ \sin(x-1)\cos(x-1)\, dx$$

$$= \frac{1}{2}\int \frac{1}{2}\left(\cos\frac{13}{4}x + \cos\frac{7}{4}x + \cos\frac{11}{4}x + \cos\frac{9}{4}x \right) dx$$

$$I = \frac{1}{2}\left(\frac{4}{13}\sin\frac{13}{4} + \frac{4}{7}\sin\frac{7}{4}x + \frac{4}{11}\sin\frac{11}{4}x + \frac{4}{9}\sin\frac{9}{4}x \right) + c$$

12) $\int \frac{1+\sin 3x}{\cos^2 3x}dx = \int \frac{1}{\cos^2 3x}dx + \int \frac{\sin 3x}{\cos^2 x}dx$

$= \frac{1}{3}\tan 3x - \frac{1}{3} . \frac{1}{\cos 3x} + c$

13) $\int \frac{(\sin x + \cos x)}{\sqrt[3]{(\sin x - \cos x)}}$ $u = \sin x - \cos x \Rightarrow du = (\cos x + \sin x)_{dx}$

$= \int \frac{dy}{\sqrt[3]{u}} = \int u^{-\frac{1}{3}} du = \frac{2}{3} u^{\frac{2}{3}} + c = \frac{2}{3}(\sin x - \cos x)^{\frac{2}{3}} + c$

14) $\int x^{n-1} \sin x^n dx$ $u = x^n \quad \Rightarrow \quad \frac{du}{n} = x^{n-1}\, dx$

$= \frac{1}{n}\int \sin u\, du = -\frac{1}{n}\cos x^n + c$

15) $\int \frac{\sin^3 x}{\sqrt[5]{\cos^3 x}} = \int \sin x \frac{1-\cos x^n x}{\sqrt[5]{\cos^3 x}} dx$

$= \int \cos^{-\frac{3}{5}} x . \sin x\, dx - \int \cos^{\frac{13}{5}} x . \sin x\, dx$

$= -\frac{5}{2}\cos^{\frac{2}{5}} x + \frac{5}{18}\cos^{\frac{18}{5}} x + c$

16) $\int \frac{\sin x}{(1+\cos x)2} dx = -\int \frac{du}{u^2} = \frac{1}{u} + c = \frac{1}{1+\cos x} + c$

17) $\int \frac{dx}{\sqrt{x}\sin^2 \sqrt{x}}$, $u = \sqrt{x} \quad \Rightarrow \quad 2du = \frac{dx}{\sqrt{x}}$

$I = 2\int \frac{du}{\sin^2 u} = -2\cot(\sqrt{x}) + c$

18) $\int \frac{x\,dx}{\cos^2 x^2}, \qquad u = x^2 \;\Rightarrow\; \frac{du}{2} = x\,dx$

$I = \frac{1}{2}\int \frac{du}{\cos^2 u} = \frac{1}{2}\tan x^2 + c$

19) $\int \sin x(1+\cos x)^5\, dx = -\int u^5 du = -\frac{(1+\cos x)^6}{6} + c$

20) $\int \sin 2x\sqrt{2+\sin^2 x}\;\; dx = \int \sqrt{u}\;\; du = \frac{2}{3}(1+\sin^2 x)\sqrt{1+\sin^2 x} + c$