فصل هشتم

روش‌های انتگرال‌گیری

تمرین صفحه 436 .

انتگرال $\int x^n Lnx\, dx$ را حل کنید.

حل) اگر فرض کنیم. $dv = x^n dx, u = Lnx$ داریم:

$$v = \frac{x^{n+1}}{n+1} \quad , \quad du = \frac{dx}{x}$$

پس

$$\mathrm{I}_n = \frac{x^{n+1}}{n+1} Lnx - \int \frac{x^n}{n+1} dx = \frac{x^{n+1}}{n+1} Lnx - \frac{x^{n+1}}{(n+1)^2} + c$$

تمرین صفحه 436 .

یک فرمول بازگشتی برای $(n>1), \mathrm{I}n = \int \cos^n x dx$ پیدا کنید و به کمک آن $\int \cos^4 x\, dx$ را محاسبه کنید.

حل) $\mathrm{I}n$ را به صورت $\mathrm{I}n = \int \cos^{n-1} x . \cos x dx$ می نویسیم.

اگر فرض کنیم $dv = \cos x\, dx, u = \cos^{n-1} x$ داریم

$du = -(n-1)\cos^{n-2} x . \sin x dx$ و $V = \sin x$ در نتیجه:

$$\mathrm{I}n = \sin x . \cos^{n-1} x + (n-1)\int \cos^{n-2} x . \sin^2 x\, dx$$

$$\Rightarrow \quad \mathrm{I}_n = \sin x . \cos^{n-1} x + (n-1)\, I_{n-2} - (n-1)\, \mathrm{I}_n$$

$$\Rightarrow \quad n\mathrm{I}_n = \sin x . \cos^{n-1} x + (n-1)\, \mathrm{I}_{n-2}$$

$$n = 2 \quad \to \quad 2\mathrm{I}_2 = \sin x . \cos x + x$$

$$n = 4 \quad \to \quad 4\mathrm{I}_4 = \sin x . \cos^3 x + 3\mathrm{I}_2$$

$$\Rightarrow \mathrm{I}_4 = \frac{1}{4}(\sin x . \cos^3 x + \frac{3}{2}(\sin x . \cos x + x))$$

تمرین صفحه ۴۳۹ .

(۱) هر یک از انتگرال های زیر را محاسبه کنید.

1) $\int e^x (f(x) + f'(x))\, dx = f(x).e^x + c$

2) $\int \frac{x\, e^x}{(1+x)^2} dx = \frac{e^x}{x+1} + c$

3) $\int Ln(a^2 + x^2)\, dx = x\ Ln(a^2 + x^2) - \int \frac{x}{a^2 + x^2} dx$

$$\Rightarrow x\, Ln(a^2 + x^2) - \frac{1}{2} Ln(a^2 + x^2) + c$$

4) $\int x^3\, e^{-x^2}\, dx = -\frac{1}{2}\int u\, e^u\, du = -\frac{1}{2}(-x^2 - 1)e^{-x^2} + c$

5) $\int x^5\, e^x\, dx = (x^5 - 5x^4 + 20\, x^3 - 60\, x^2 + 120\, x - 120)e^x$

6) $\mathrm{I} = \int \sin(Lnx)\, dx$

$u = \sin(Lnx) \quad , dv = dx \Rightarrow du = \frac{1}{x}\cos(Lnx) \quad , v = x$

$\mathrm{I} = x\sin(Lnx) - \int \cos(Lnx)\, dx$

$u = \cos(Lnx) \quad , dv = dx \Rightarrow du = -\frac{1}{x}\sin(Lnx)\, , v = x$

$\mathrm{I} = x\sin(Lnx) - x\cos(Lnx) - \mathrm{I}$

$\mathrm{I} = \frac{x}{2}(\sin(Lnx) - \cos(Lnx)) + c$

7) $\mathrm{I} = \int x\ tg^{-1}x\ dx$

$u = tg^{-1}x\, , dv = x\ dx \Rightarrow dv = \frac{1}{1+x^2} dx \quad , \quad v = \frac{x^2}{2}$

$\mathrm{I} = \frac{x^2}{2} tg^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1} dx = \frac{x^2}{2} tg^{-1}x - \frac{1}{2}x + \frac{1}{2} tg^{-1}x + c$

8) $\int \sin^{-1}\sqrt{x}\ dx \qquad t^2 = x \Rightarrow 2tdt = dx$

$\mathrm{I} = 2\int t\sin^{-1} t dt = t^2 \sin^{-1} t - \int \frac{t^2}{\sqrt{1-t^2}} dt$

$= t^2 \sin^{-1} t + \int \frac{1 - t^2 + 1}{\sqrt{1-t^2}} dt$

$= t^2 \sin^{-1} t + \sin^{-1} t + \int \sqrt{1-t^2}\ dt$

$\int \sqrt{1-t^2}\ dt = \int \cos^2 q\ d q = \frac{1}{2} t + \frac{1}{4}\cos 2q$

$\Rightarrow \mathrm{I} = x\sin^{-1}\sqrt{x} + \sin^{-1}\sqrt{x} + \frac{1}{2}\cos^{-1}\sqrt{x} + \frac{1}{4}\cos 2(\cos^{-1}\sqrt{x})$

9) $I=\int (x^3+x)chx\ dx$

$I=(x^3+x)shx-(3x^2+1)chx+6x\ shx-6chx$

10) $I=\int Ln(x+\sqrt{1+x^2})dx$

$u=Ln(x+\sqrt{1+x^2}),dv=dx\Rightarrow du=\frac{1}{\sqrt{1+x^2}},v=x$

$I=x\ Ln(x+\sqrt{1+x^2})-\int\frac{x}{\sqrt{1+x^2}}=x\ Ln(x+\sqrt{1+x^2})-\sqrt{1+x^2}+c$

11) $I=\int x\ Ln(\frac{1+x}{1-x})dx$

$u=Ln(\frac{1+x}{1-x}),dv=dx\Rightarrow du=\frac{2}{1-x^2}dx \qquad ,v=x$

$I=x\ Ln(\frac{1+x}{1-x})-\int\frac{2x}{1-x^2}dx=x\ Ln(\frac{1+x}{1-x})+Ln(1-x^2)+c$

12) $I=\int x^2\ Ln(\frac{1+x}{1-x})dx$

$u=Ln(\frac{1-x}{1+x}) \quad , \qquad dv=x^2\,dx \qquad \Rightarrow du=-\frac{2}{1-x^2}dx \qquad ,v=\frac{x^3}{3}dx$

$I=\frac{x^3}{3}Ln(\frac{1-x}{1+x})+\frac{1}{3}\int x^3 1-x^2\,dx$

$=\frac{x^3}{3}Lx(\frac{1-x}{1+x})-\frac{1}{3}\int(x+\frac{x}{x^2-1})dx$

$=\frac{x^3}{3}Lx(\frac{1-x}{1+x})-\frac{1}{3}x^2-\frac{1}{6}Ln(x^2-1)+c$

13) $I=\int e^{\sqrt{x}}\ dx$

$u^2=x\Rightarrow 2udu=dx$

$I=2\int u\,e^u\,du=2(u-1)e^u+c=2(\sqrt{x}-1)e^{\sqrt{x}}+c$

14) $I=\int\frac{tg^{-1}e^x}{e^x}dx$

$u=tg^{-1}e^x \qquad ,dv=\frac{dx}{e^x}\Rightarrow du=\frac{e^x}{1+e^{2x}}dx,v=-\frac{1}{e^x}$

$I=-\frac{tg^{-1}e^x}{e^x}+\int\frac{dx}{1+e^{2x}}=\frac{tg^{-1}e^x}{e^x}+\int\frac{du}{u(1+u^2)}$

$=-\frac{tg^{-1}e^x}{e^x}+Ln\ e^x+\frac{1}{2}\ Ln(1+e^{2x})-2tg^{-1}e^x+c$

15) $\mathrm{I}=\int(\sin^{-1}x)^2\,dx$

$u=(\sin^{-1}x)^2 \quad , dv=dx \quad \Rightarrow du=\frac{2}{\sqrt{1-x^2}}\sin^{-1}x\;dx\;,\;v=x$

$\mathrm{I}=x(\sin^{-1}x)^2-\int\frac{2}{\sqrt{1-x^2}}\sin^{-1}x\;dx$

$=x(\sin^{-1}x)^2+2(\sqrt{1-x^2})\sin^{-1}x-2x+c$

16) $\mathrm{I}=\int(\frac{Lnx}{x})^2\,dx$

$u=(\frac{Lnx}{x})^2 \quad , \quad dv=dx \quad \Rightarrow du=2(\frac{1-Lnx}{x^2})(\frac{Lnx}{x})\,dx\,,v=x$

$\mathrm{I}=x(\frac{Lnx}{x})^2-2\int\frac{Lnx-(Lnx)^2}{\ }x^2\,dx$

$=x(\frac{Lnx}{x})^2-2\int\frac{Lnx}{x^2}dx+2\int(\frac{Lnx}{x})^2\,dx$

$-\mathrm{I}=x(\frac{Lnx}{x})^2+2\int\frac{1-Lnx-1}{\ }x^2\,dx$

$\mathrm{I}=-x(\frac{Lnx}{x})^2-2(\int(\frac{Lnx}{x})'dx-\int\frac{1}{x^2}dx)+c$

$=-x(\frac{Lnx}{x})^2-2(\frac{Lnx}{x})+\frac{1}{x}+c$

17) $\mathrm{I}=\int\frac{Ln(1+x)}{2(1+x)}dx=\frac{1}{2}\int u\;du$

$=\frac{1}{4}(Ln(1+x))^2+c$

18) $\mathrm{I}=\int_0^{\frac{p^2}{2}}\cos\sqrt{2x}\,dx$

$u=\sqrt{2x} \quad \Rightarrow \quad 2udu=2dx$

$\mathrm{I}=\int_0^{p}u\cos u\;du=u\sin u+\cos u+c$

19) $\mathrm{I}=\int_1^4\sec^{-1}\sqrt{x}\;dx \quad , \quad u=\sqrt{x}$

$2u\,du=dx$

$\mathrm{I}=2\int_1^2u\sec^{-1}u\;du=2\int_1^2u\cos^{-1}(\frac{1}{u})\,du$

$t=\cos^{-1}(\frac{1}{u})\Rightarrow dt=-\frac{1}{u^2}.\frac{-1}{\sqrt{1-\frac{1}{u^2}}}\;du=\frac{1}{u\sqrt{u^2-1}}du$

$$I = 2(\frac{u^2}{2}\cos^{-1}(\frac{1}{u})) - \int \frac{u}{\sqrt{u^2-1}}\,du$$

$$= u^2\cos^{-1}(\frac{1}{u}) - \sqrt{u^2-1}\Big|_1^2 = 4\times\frac{p}{3} - \sqrt{3}$$

20) $$I = \int_{\frac{p}{4}}^{\frac{3p}{4}} x.\cot x.\csc x\,dx = -\int_{\frac{p}{4}}^{\frac{3p}{4}} x(\csc x)'\,dx$$

$$u = x \quad , \quad dv = (\csc x)'dx \Rightarrow du = dx \;,\; v = \csc x$$

$$I = x\csc x\Big|_{\frac{p}{4}}^{\frac{3p}{4}} - \int_{\frac{p}{4}}^{\frac{3p}{4}} \csc x\;dx$$

$$\int \csc x\;dx = \int \frac{dx}{\sin x} = \int \frac{\sin x}{1-\cos^2 x}\;dx = \frac{1}{2}Ln\left|\frac{1-\cos x}{1+\cos x}\right|$$

$$I = (\frac{3p}{4}\times\sqrt{2} - \frac{p}{4}\times\sqrt{2}) - \frac{1}{2}(Ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| - Ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|)$$

21) $$I = \int_0^{\frac{p^2}{4}} \sin\sqrt{x}\;dx \qquad\qquad u^2 = x \Rightarrow 2u\,du = dx$$

$$I = 2\int_0^{\frac{p}{2}} u\;\sin u\,du = 2(-u\;\cos u + \sin u)\Big|_0^{\frac{p}{2}}$$

تمرین صفحه ۴۴۴ .

هر یک از انتگرال های زیر را حل کنید.

1) $\int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}\,dx$

$f(x) = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^2}$

$\Rightarrow (Ax+B)(x^4+2x^2+1) + (Cx+D)(x^2+1) + Ex + F$

$= x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4$

$\Rightarrow A = 1 \quad , \quad B = -1 \quad , \quad 2A + C = 4 \Rightarrow C = 2$

$2B$

2) $\int \frac{(\sec^2 x + 1).\sec^2 x}{1 + tg^2 x}$

$u = tgx \Rightarrow du = \sec^2 x dx \;\; , \;\; u^2 = \sec^2 x - 1$

$I = \int \frac{(u^2+2)du}{1+u^3}$

$\frac{u^2+2}{1+u^3} = \frac{A}{1+u} + \frac{Bu+c}{1+u+u^2}$

$\Rightarrow (A+B)u^2 + (A+B+C)u + A + C = u^2 + 2$

$A + B = 1$

$A + B + C = 0 \Rightarrow C = -1 \quad , \quad A = 3 \quad , \quad B = -2$

$A + C = 2$

$\Rightarrow I = \int \frac{3}{u+1}du - \int \frac{2u+1}{u^2+u+1}du = 3\ln(u+1) - \ln(u^2+u+1) + c$

$I = 3\ln(tgx+1) - \ln(tg^2 x + tgx + 1) + c$

3) $I=\int\frac{x^2+2x-1}{27x^3-1}dx$

$$\frac{x^2+2x-1}{(3x-1)(9x^2+3x+1)}=\frac{A}{3x-1}+\frac{Bx+C}{9x^2+3x+1}$$

$$\Rightarrow (9A+3B)x^2+(3A-B+3C)x+A=x^2+2x-1$$

$$\Rightarrow 9A+3B=1 \quad , \quad 3A-+3C=2 \quad ,A-C=-1$$

$$A=C-1 \quad , \quad 9C-9+3B=1$$

$$3C-3-B+3C=2$$

$$\Rightarrow \begin{cases}9C+3B=10\\ 6C-B=5\end{cases} \quad \Rightarrow 27C=25 \Rightarrow C=\frac{25}{27}$$

$$A=\frac{-2}{27} \quad , \quad B=\frac{75}{9}-\frac{45}{9}=\frac{10}{3}$$

$$\Rightarrow -\frac{2}{27}\int\frac{dx}{3x-1}=-\frac{2}{81}\ln(3x-1)$$

$$\frac{\frac{10}{3}x-\frac{25}{27}}{9x^2+3x+1}=\frac{1}{27}.\frac{90x-25}{9x^2+3x+1}$$

$$I=-\frac{2}{81}\ln(3x-1)+\frac{1}{27}\int\frac{90x-25}{9x^2+3x+1}dx$$

4) $I=\int\frac{dx}{x^3+x^2+x}=\int\frac{dx}{x(x^2+x+1)}$

$$\frac{1}{x(x^2+x+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+x+1}$$

$$A=1 \quad , \quad B=-1 \, , \quad C=-1$$

$$I=\int\frac{dx}{x}-\int\frac{x+1}{x^2+x+1}=\ln x-\frac{1}{2}\ln(x^2+x+1)+\frac{2}{\sqrt{3}}tg^{-1}\left(\frac{(x+\frac{1}{2})}{\frac{\sqrt{3}}{2}}\right)$$

5) $$\int \frac{dq}{\cos q\,(1+\sin q)}$$

$$u=\sin q \Rightarrow \frac{du}{\cos q}=dq$$

$$I=\int \frac{du}{(1-u)^2(1+u)}=\int \frac{du}{(1+u)^2(1-u)}$$

$$\frac{1}{(1+u)^2(1-u)}=\frac{A}{1+u}+\frac{B}{(1+u)^2}+\frac{C}{1-u}$$

$$C=\frac{1}{4}\ ,\qquad B=\frac{1}{2}\ ,\qquad A=\frac{1}{4}$$

$$I=\frac{1}{4}\int \frac{du}{1+u}+\frac{1}{2}\int \frac{du}{(1+u)^2}-\frac{1}{4}\int \frac{du}{u-1}$$

$$I=\frac{1}{4}\ln(1+\sin q)-\frac{1}{2(1+\sin q)}-\frac{1}{4}\ln(\sin q-1)+C$$

6) $$\int \frac{dq}{\sin(1+\sin q)}=\int \frac{dq}{\sin q}-\int \frac{dq}{1+\sin q}$$

$$=\frac{1}{2}\ln\left|\frac{\sin q-1}{\sin q+1}\right|-\int \frac{1-\sin q}{\cos^2 q}dq$$

$$=\frac{1}{2}\ln\left|\frac{\sin q-1}{\sin q+1}\right|-tgq=\frac{1}{\cos q}+C$$

7) $\int \frac{dt}{(1+t)(1+t^2)^2}$

$$\frac{1}{(1+t)(1+t^2)^2} = \frac{A}{1+t} + \frac{Bt+C}{t^2+1} + \frac{Dt+E}{(t^2+1)^2}$$

$$A(t^2+1)^2 + (Bt+C)(t^2+1) + dt + E = 1$$

$$A = \frac{1}{4} \quad , \qquad B = -\frac{1}{4}$$

$$A + C + E = 1 \ \Rightarrow \qquad C + E = \frac{3}{4}$$

8) $\int \frac{x^2+1}{x^3+8} dx$

$$\frac{x^2+1}{x^3+8} = \frac{x^2+1}{(x^2+2)(x^2+2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+4}$$

$$\Rightarrow (A+B)x^2 + (2A+2B+2C)x + 4A + 2C = x^2 + 1$$

$$A + B = 1$$

$$\Rightarrow C = -1 \qquad , \qquad A = \frac{3}{4} \quad , \qquad B = \frac{1}{4}$$

$$A + B + C = 0$$

$$4A + 2C = 1$$

$$I = \frac{1}{4}\int \frac{dx}{x+2} + \frac{1}{4}\int \frac{3x+1}{x^2+2x+4} dx$$

$$= \frac{1}{4}\ln(x+2) + \frac{1}{4}\int \frac{2x+2+(x-1)}{x^2+2x+4} dx$$

$$= \frac{1}{4}\ln(x+2) + \frac{1}{4}\ln(x^2+2x+4) + \frac{1}{4}\int \frac{(x-1)}{x^2+2x+4}$$

$$\int \frac{(x-1)}{(x+1)^2+3} dx = \frac{1}{2}\ln((x+1)^2+3) - \frac{2}{\sqrt{3}} tg^{-1}(\frac{x+1}{\sqrt{3}})$$

9) $\int \frac{dx}{x^4-3x^3} = \int \frac{dx}{x^3(x-3)}$

$$\frac{dx}{x^3(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$$

$$C = -\frac{1}{3} \quad , \quad D = 1 \quad , \quad A = -1 \; , \quad B = \frac{4}{3}$$

$$I = -\ln x - \frac{4}{3}.\frac{1}{x} - \frac{1}{6}.\frac{1}{x^2} + \ln(x-3) + C$$

10) $\int \frac{x^2+2}{4x^5+4x^3+x} = \int \frac{x^2+2}{x(2x^2+1)^2}$

$$\frac{x^2+2}{x(2x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$$

$$A = 2 \quad , \quad B = -4 \; , \; \ldots$$

11) $\int \frac{xdx}{(x^2-x+1)^2}$

$$\frac{x}{(x^2-x+1)^2} = \frac{Ax+B}{x^2-x+1} = \frac{Cx+D}{(x^2-x+1)^2}$$

$$A = 1 \quad , \quad B + D = 0 \quad , \quad C = 0$$

$$3B + D = 2 \;\; \Rightarrow \;\; B = 1 \; , \quad D = -1$$

12) $\int_0^1 \frac{dx}{x^3+2x^2+x+2}$

13) $\int_0^4 \frac{x^2dx}{2x^3+9x^2+12x+4}$

14) $\int_{-1}^0 \frac{x^2dx}{(2x^2+2x+1)^2}$

تمرین صفحهٔ 449.

هر یک از انتگرال های زیر را حل کنید.

1) $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\sin^{-1}(2x) + C$

2) $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

$x = 3\sin q \quad \Rightarrow dx = 3\cos q\, dq$

$I = \int \frac{9\sin^2 q . 3\cos q dq}{3\cos q} = \frac{9}{2}q - \frac{9}{4}\sin 2q$

3) $\int \frac{x^3 dx}{\sqrt{9-x^2}} = 27\int \sin^3 q dq = 27\int \sin q - \cos^2 q \sin q$

$= -27\cos q + 9\cos^3 q + c$

4) $\int \frac{dx}{x\sqrt{9-x^2}} = \frac{1}{3}\int \frac{dq}{\sin q} = -\frac{1}{6}\ln\left|\frac{\sin q + 1}{\sin q - 1}\right| + 2$

5) $\int x^2\sqrt{9-x^2}\, dx$

$x = 3\sin q \;\Rightarrow\; I = 81\int \sin^2 q . \cos^2 q dq$

$I = \frac{81}{4}\int \sin^2 2q\, dq = \frac{81}{8}q - \frac{81}{32}\sin 4q + C$

6) $\int \frac{x+1}{\sqrt{9-x^2}}dx = \int (\sin q + 1)dq = -\cos q + q$

7) $\int \frac{x^3}{\sqrt{9+x^2}}dx \qquad x = 3tgq \;\Rightarrow\; dx = 3\sec^2 q dq$

$I = 27\int tg^3 q\, dq = 27\int tgq(\sec^2 q - 1)dq$

$= \frac{27}{2}tg^2 q + 27\ln(\cos q) + C$

8) $I = \int \frac{\sqrt{1-x^2}}{x^2} dx = \int \frac{\cos^2 q}{\sin^2 q} dq = -\cot q - q + C$

9) $I = \int_{-\ln 2}^{0} e^x \sqrt{1-e^{2x}} dx$

$u = e^x \Rightarrow I = \int_{\frac{1}{2}}^{1} \sqrt{1-u^2} du = \int_{\frac{p}{6}}^{\frac{p}{2}} \cos^2 q \, dq$

$= (\frac{1}{2} q + \frac{1}{4} \sin 2q)\Big|_{\frac{p}{6}}^{\frac{p}{2}} = \frac{p}{4} - (\frac{p}{12} + \frac{\sqrt{3}}{8})$

10) $\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{dx}{((x+1)^2+1)^2} = \int \frac{du}{(u^2+1)^2}$

$\int \frac{\sec^2 q}{\sec^4 q} dq = \int \cos^2 q dq = \frac{1}{2} q + \frac{1}{4} \sin 2q + C$

11) $\int \frac{dx}{(1+2x^2)^{\frac{5}{2}}}$, $\sqrt{2} x = tg q$

$I = \frac{1}{\sqrt{2}} \int \frac{\sec^2 q}{\sec^5 q} dq = \frac{1}{\sqrt{2}} \int \cos^3 q \, dq$

$= \frac{1}{\sqrt{2}} \int (\cos q - \sin^2 q \cos q) dq = \frac{1}{\sqrt{2}} \sin q - \frac{\sin^3 q}{3\sqrt{2}} + C$

12) $$\int \frac{dq}{2+\sin q} = \int \frac{\frac{2dt}{1+t^2}}{2+\frac{2t}{1+t^2}} dt = \int \frac{dt}{t^2+t+1}$$

$$= \frac{2}{\sqrt{3}} tg^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

13) $$\int_0^{\frac{p}{2}} \frac{dq}{1+\sin q+\cos q} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int_0^1 \frac{2dt}{2+2t}$$

$$= \ln(1+t)\Big|_0^1 = \ln 2$$

14) $$\int \frac{dq}{3+2\cos q} = \int \frac{\frac{2dt}{1+t^2}}{3+2+\frac{1-t^2}{1+t^2}} dt$$

$$= \int \frac{2dt}{t^2+5} = \frac{2}{\sqrt{5}} tg^{-1}\left(\frac{t}{\sqrt{5}}\right) + C$$

15) $\int \sec^3 x dx = \int \frac{dx}{\cos^2 x} = \int \frac{du}{(1-u^2)^2}$

$u = \sin x \;\Rightarrow\; dx = \frac{du}{\cos x}$

$\frac{1}{(1-u)^2(1+u)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}$

$D = \frac{1}{4}$, $B = \frac{1}{4}$, $A + C = 0$, $A - C = -\frac{1}{2}$

$A = -\frac{1}{4}$, $C = \frac{1}{4}$

$I = -\frac{1}{4}\ln|\sin x + 1| - \frac{1}{2}.\frac{1}{1+\sin x} + \frac{1}{4}\ln|\sin x - 1| - \frac{1}{4}.\frac{1}{\sin x - 1}$

16) $\int \csc x dx = \int \frac{dx}{\sin x} = \int \frac{dt}{t} = \ln\left|tg\frac{x}{2}\right| + C$

17) $I = \int x \sin^{-1} x dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2 q\, dq$

$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}(\sin^{-1} x) + \frac{1}{8}\sin 2(\sin^{-1} x) + C$

18) $\int \frac{dx}{1 - \sin x + \cos x} = \int \frac{\frac{2dt}{1+t^2}}{1 - \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \int \frac{dt}{1-t}$

$= -\ln\left|tg\frac{x}{2} - 1\right| + C$

19) $\int_0^p \frac{\sin x}{4+\cos^2 x}dx = -\int_1^{-1}\frac{du}{4+u^2} = \frac{1}{2}tg^{-1}u\Big|_{-1}^{1} = \frac{1}{2}\times\frac{p}{2} = \frac{p}{4}$

20) $\int \frac{dx}{x^4\sqrt{x^{2-1}}} = \int \frac{\sec q\,.tgq}{\sec^4 q\,.tgq}dq = \int \cos^3 q dq$

$= \int (1-\sin^2 q)\cos q dq = \sin q - \frac{\sin^3 q}{3} + C$

21) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C$

22) $\int \frac{dx}{4x^2+12x+13} = \int \frac{dx}{(2x+3)^2+4} = \frac{1}{4}tg^{-1}(\frac{2x+3}{2})$